# Esercitazione

I parte: limiti con Taylor
II parte: Calcolo di integrali

## Esercizi sui limiti con la formula di Taylor.

1) $$\lim_{x\to 0} \frac{e^{-x^2} - 2\cos x + 1}{x^2 \log(1+3x^2)} \qquad \left(\frac{0}{0}\right)$$

$$\log(1+x) = x + o(x)$$
$$\log(1+3x^2) = 3x^2 + o(x^2)$$

Denom.: $x^2 \log(1+3x^2) =$
$$= x^2 \left(3x^2 + o(x^2)\right)$$
$$= 3x^4 + o(x^4)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 + o(x^4)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$e^{-x^2} - 2\cos x + 1 = 1 - x^2 + \frac{1}{2}x^4 + o(x^4) +$$
$$-2\left(1 - \frac{1}{2}x^2 + \frac{x^4}{24} + o(x^4)\right) + 1$$
$$= \cancel{1} - \cancel{x^2} + \frac{1}{2}x^4 - \cancel{2} + \cancel{x^2} - \frac{1}{12}x^4 + \cancel{1} + o(x^4)$$
$$= \frac{5}{12}x^4 + o(x^4)$$

$$\Rightarrow \lim_{x\to 0} \frac{e^{-x^2} - 2\cos x + 1}{x^2 \log(1+3x^2)} = \lim_{x\to 0} \frac{\frac{5}{12}x^4 + o(x^4)}{3x^4 + o(x^4)}$$

$x \to 0 \quad x^2 \log(1+3x^2) \qquad x \to 0 \quad 3x^4 + o(x^4)$

$$= \lim_{x \to 0} \frac{\cancel{x^4}\left(\frac{5}{12} + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0}\right)}{\cancel{x^4}\left(3 + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0}\right)} = \frac{5}{36}.$$

2) $$\lim_{x \to 0} \left[\left(\frac{\sin x}{x} - e^{x^2}\right)\frac{1}{x \sin x}\right] \qquad \left(\frac{0}{0}\right)$$

Denom.: $x \sin x = x\,(x + o(x))$

$= x^2 + o(x^2)$

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$$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$= 1 - \frac{x^2}{6} + o(x^2)$$

$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$

$e^{x^2} = 1 + x^2 + o(x^2)$

$$\frac{\sin x}{x} - e^{x^2} = \cancel{1} - \frac{x^2}{6} + o(x^2) - (\cancel{1} + x^2 + o(x^2))$$

$$= -\frac{7}{6}x^2 + o(x^2)$$

$$\lim_{x \to 0} \left(\frac{\sin x}{x} - e^{x^2}\right)\frac{1}{x \sin x} = \lim_{x \to 0} \frac{-\frac{7}{6}x^2 + o(x^2)}{x^2 + o(x^2)}$$

$$= \ldots \quad \cancel{x^2}\left(-\frac{7}{6} + \underbrace{o(x^2)}_{\to 0}\right) \ldots$$

$$\left( = \lim_{x \to 0} \frac{\cancel{x^2}\left(-\frac{7}{6} + \frac{o(x^2)}{x^2}\right)}{\cancel{x^2}\left(1 + \underbrace{\frac{o(x^2)}{x^2}}_{0}\right)} \right) = -\frac{7}{6}.$$

3) $$\lim_{x \to +\infty} \frac{x^2\left(e^{\frac{1}{x^2}} - \cos\frac{1}{x} - \frac{3}{2x^2}\right)}{\sin^2\left(\frac{1}{x}\right)}$$

$\sin x = x + o(x)$ per $x \to 0$

$\sin\frac{1}{x} = \frac{1}{x} + o\left(\frac{1}{x}\right)$ per $x \to +\infty$

$$\sin^2\frac{1}{x} = \left(\frac{1}{x} + o\left(\frac{1}{x}\right)\right)^2$$

$$= \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$$

$$e^{\frac{1}{x^2}} - \cos\frac{1}{x} - \frac{3}{2x^2} =$$

$$= 1 + \frac{1}{x^2} + \frac{1}{2x^4} + o\left(\frac{1}{x^4}\right) +$$

$$-\left(1 - \frac{1}{2x^2} + \frac{1}{24 \cdot x^4} + o\left(\frac{1}{x^4}\right)\right) - \frac{3}{2x^2}$$

$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$

$$= \cancel{1} + \cancel{\frac{1}{x^2}} + \frac{1}{2x^4} - \cancel{1} + \cancel{\frac{1}{2x^2}} - \frac{1}{24x^4} - \cancel{\frac{3}{2x^2}} + o\left(\frac{1}{x^4}\right)$$

$$= \frac{11}{24} \cdot \frac{1}{x^4} + o\left(\frac{1}{x^4}\right)$$

$$\Rightarrow \lim_{x \to +\infty} \frac{x^2\left(e^{\frac{1}{x^2}} - \cos\frac{1}{x} - \frac{3}{2x^2}\right)}{\sin^2\left(\frac{1}{x}\right)} = \lim_{x \to +\infty} \frac{x^2\left(\frac{11}{24} \cdot \frac{1}{x^4} + o\left(\frac{1}{x^4}\right)\right)}{\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)}$$

$$\sin^2\left(\frac{1}{x}\right) \quad \underset{x\to+\infty}{=} \quad \frac{1}{x^2}+o\left(\frac{1}{x^2}\right)$$

$$= \lim_{x\to+\infty} \frac{\frac{11}{24}\cdot\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)} = \frac{11}{24}.$$

4) $$\lim_{x\to 0} \frac{\sqrt[3]{1+x}-e^x}{\sqrt[3]{x^2}\left(e^{\sqrt[3]{x}}-1\right)} \qquad \left(\frac{0}{0}\right)$$

Denom. $\sqrt[3]{x^2}\left(e^{\sqrt[3]{x}}-1\right) =$

$$e^x = 1+x+\frac{x^2}{2}+o(x^2)$$

$$e^x-1 = \underbrace{x+o(x)}$$

$$= \sqrt[3]{x^2}\left(\underbrace{\sqrt[3]{x}+o\left(\sqrt[3]{x}\right)}\right)$$

$$= \underline{x}^{\frac{2}{3}}\left(\underline{x}^{\frac{1}{3}}+o\left(x^{1/3}\right)\right)$$

$$= x+\underline{o}(x)$$

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$$\sqrt[3]{1+x} = 1+\frac{1}{3}x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2+o(x^2)$$

$$\sqrt[3]{1+x}-e^x = \cancel{1}+\frac{1}{3}x+o(x)-\left(\cancel{1}+x+o(x)\right)$$

$$= -\frac{2}{3}x+o(x)$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt[3]{1+x}-e^x}{\sqrt[3]{x^2}\left(e^{\sqrt[3]{x}}-1\right)} = \lim_{x\to 0}\frac{-\frac{2}{3}x+o(x)}{x+o(x)} = -\frac{2}{3}.$$

$$\lim_{x\to 0} \frac{\ldots}{\sqrt[3]{x^2}\left(e^{\sqrt[3]{x}}-1\right)} = \lim_{x\to 0} \frac{\ldots}{x+o(x)} = -\frac{2}{3}.$$

$$\left(\lim_{x\to 0} \frac{\cancel{x}\left(-\frac{2}{3}+\frac{o(x)}{x}\right)}{\cancel{x}\left(1+\frac{o(x)}{x}\right)}\right)$$

$\frac{o(x)}{x} \to 0$

5) $$\lim_{x\to 0^+} \frac{2\cos\left(e^x-1\right)+\sin\left(x^2+x^3\right)-2}{x^3} \quad \left(\frac{0}{0}\right)$$

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)$$

$$\cos x = 1-\frac{1}{2}x^2+\frac{x^4}{24}+o(x^4)$$

$$\cos\left(e^x-1\right) =$$

$$= \cos\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\right)$$

$$= 1-\frac{1}{2}\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\right)^2+\frac{1}{24}\underbrace{\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\right)^4}_{x^4+o(x^4)}$$

$$= 1-\frac{1}{2}\left(x^2+\cancel{2}\cdot\frac{1}{\cancel{2}}x^3+\frac{1}{4}x^4+\cancel{2}\cdot\frac{1}{\cancel{6}_3}x^4+o(x^4)\right)+$$

$$+\frac{1}{24}x^4+o(x^4).$$

$$= 1-\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{4}x^4+o(x^4)$$

A)

$$\sin x = x-\frac{x^3}{6}+o(x^3)$$

$$\sin\left(x^2+x^3\right) = x^2+x^3-\frac{\left(x^2+x^3\right)^3}{6}+$$

$$\sin(x^2+x^3) = x^2+x^3 - \frac{(x^2+x^3)^3}{6} + o(x^6)$$

$$= x^2+x^3+o(x^6)$$

$$\lim_{x\to 0} \frac{2\cos(e^x-1)+\sin(x^2+x^3)-2}{x^3} =$$

$$= \lim_{x\to 0} \frac{2\left(1-\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{4}x^4+o(x^4)\right)+x^2+x^3+o(x^6)-2}{x^3}$$

$$= \lim_{x\to 0} \frac{\cancel{2}-\cancel{x^2}-\cancel{x^3}-\frac{1}{2}x^4+\cancel{x^2}+\cancel{x^3}-\cancel{2}+o(x^4)}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}x^4+o(x^4)}{x^3} = \lim_{x\to 0} \frac{x^{\cancel{4}}\left(-\frac{1}{2}+\underbrace{\frac{o(x^4)}{x^4}}_{\to 0}\right)}{\cancel{x^3}}$$

$$= \lim_{x\to 0} -\frac{1}{2}x = 0.$$

Per es.

$$\lim_{x\to 0} \frac{\frac{\log(1+x)}{x}-\frac{1}{\sqrt{1+x}}}{x^2} \quad \left(=-\frac{1}{24}\right).$$

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Esercitaz.: 2ª parte

Calcolo di integrali immediati e "quasi immediati".

Esercizi:

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1) $\int x^7 \, dx = \frac{x^8}{8} + c$

2) $\int \sqrt{x^3} \, dx = \int x^{\frac{3}{2}} \, dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= \frac{2}{5} \cdot \sqrt{x^5} + c$

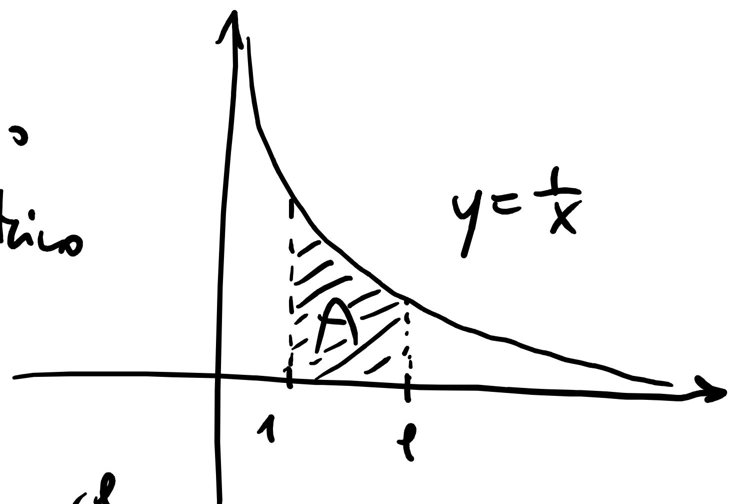
3) $\int (x^3 + 2x^5 - 5) \, dx = \int x^3 + 2\int x^5 - 5\int 1 \, dx$

$= \frac{x^4}{4} + \cancel{2}^{1} \cdot \frac{x^6}{\cancel{6}_{3}} - 5x + c$

Integrale definito

4) $\int_1^e \frac{1}{x} \, dx = \ln|x| \Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1.$

Significato geometrico




$A = \int_1^e \frac{1}{x} \, dx$

Integrali "quasi-immediati".

1) $\int [f(x)]^{\alpha} \cdot f'(x) \, dx = \frac{[f(x)]^{\alpha+1}}{\alpha+1} + c$ $\quad \left( \int x^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + c \right)$ $\alpha \neq -1$

Infatti $D\left(\frac{[f(x)]^{\alpha+1}}{\alpha+1}\right) = \frac{1}{\cancel{\alpha+1}} \cdot \cancel{(\alpha+1)}[f(x)]^{\alpha} \cdot f'(x)$

$= [f(x)]^{\alpha} \cdot f'(x)$ .

2) $\int \frac{1}{f(x)} \cdot f'(x)\, dx = \log|f(x)| + c$ $\left(\int \frac{1}{x} = \log|x| + c\right)$

3) $\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + c$ $\left(\int e^{x} = e^{x} + c\right)$

4) $\int \sin(f(x)) \cdot f'(x)\, dx = -\cos f(x) + c$ $\left(\int \sin x = -\cos x + c\right)$

5) $\int \cos(f(x)) \cdot f'(x)\, dx = \sin f(x) + c$

6) $\int \frac{1}{\cos^2(f(x))} \cdot f'(x)\, dx = \operatorname{tg} f(x) + c$ $\left(\int \frac{1}{\cos^2 x} = \operatorname{tg} x + c\right)$

7) $\int \frac{1}{1+[f(x)]^2} \cdot f'(x)\, dx = \operatorname{arctg} f(x) + c$ $\left(\int \frac{1}{1+x^2} = \operatorname{arctg} x + c\right)$

8) $\int \frac{1}{\sqrt{1-[f(x)]^2}} \cdot f'(x)\, dx = \arcsin f(x) + c$

Esempi:

1) $\int \frac{2x}{\phantom{x}}\, dx = \ln|x^2+1| + c$

1) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln|x^2+1| + c$

2) $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + c$

3) $\displaystyle\int e^{x^3}\cdot x^2\,dx = \frac{1}{3}\int e^{x^3}\cdot 3x^2\,dx = \frac{1}{3}e^{x^3} + c$

$\displaystyle\left(\int e^{f(x)}\cdot f'(x) = e^{f(x)} + c\right)$

4) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \operatorname{arctg} e^x + c$

$\displaystyle\left(\int \frac{1}{1+f(x)^2}\cdot f'(x)\,dx = \operatorname{arctg} f(x) + c\right)$

$\shortparallel$

$\displaystyle\int \frac{e^x}{1+(e^x)^2}\,dx$

$e^{2x} = (e^x)^2$ !

5) $\displaystyle\int x\cdot\sqrt[3]{x^2+1}\,dx = \frac{1}{2}\int 2x\,(x^2+1)^{\frac{1}{3}}\,dx =$

$\displaystyle= \frac{1}{2}\cdot\frac{(x^2+1)^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c$

$\displaystyle\left(\int [f(x)]^{\alpha}\cdot f'(x) = \frac{[f(x)]^{\alpha+1}}{\alpha+1} + c\right)$

$\displaystyle= \frac{1}{2}\,\frac{(x^2+1)^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{1}{2}\cdot\frac{3}{4}\cdot\sqrt[3]{(x^2+1)^4} + c$

$\displaystyle= \frac{3}{8}\sqrt[3]{(x^2+1)^4} + c$ .

6) $\displaystyle\int_0^{\frac{\pi}{5}} \sin(5x)\, dx = \frac{1}{5}\int_0^{\frac{\pi}{5}} 5 \sin(5x)\, dx = -\frac{1}{5}\cos 5x \Big|_0^{\frac{\pi}{5}}$

$$= -\frac{1}{5}\overbrace{\cos\pi}^{=-1} + \frac{1}{5}\underbrace{\cos 0}_{=1}$$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$$